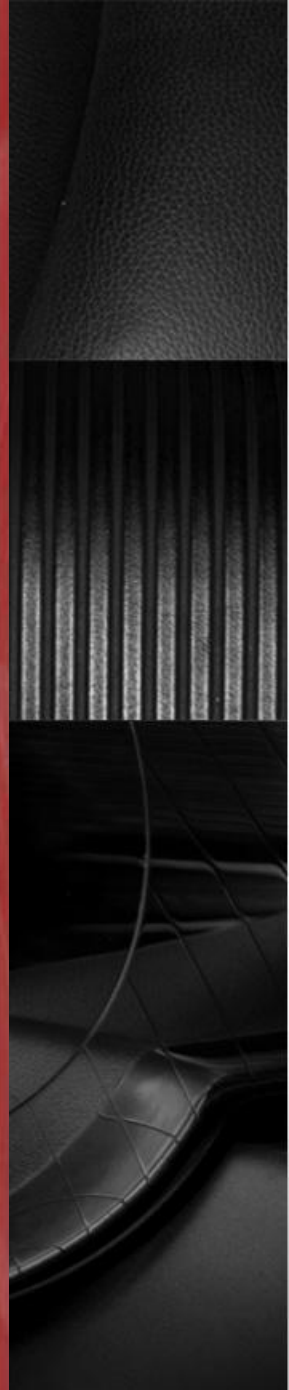


Introduction to the Lambda Calculus

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Leibniz (1646-1716)

- “Create a universal language in which all possible problems can be stated”
- “Find a decision method to solve all problems stated in the universal language”

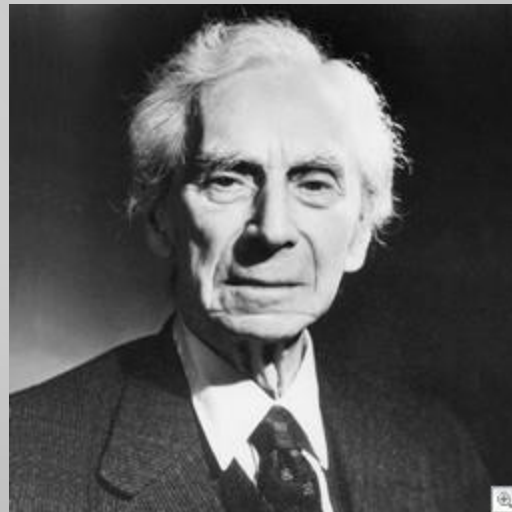


The Language

- Set theory coupled with first order logic
 - A formal logical system used in mathematics, philosophy, linguistics, and computer science
 - Many names: first-order predicate calculus, the lower predicate calculus, quantification theory, and predicate logic
 - Different from propositional logic by its use of quantifiers
 - Distinguished from higher-order logics in that quantification is allowed only over atomic entities (individuals but not sets)

- Frege and Russell
- Zermelo
- early 1900's
- Russell paradox 1901 ends naïve sets:

$$R = \{x | x \notin x\} \text{ then } R \in R \Leftrightarrow R \notin R$$



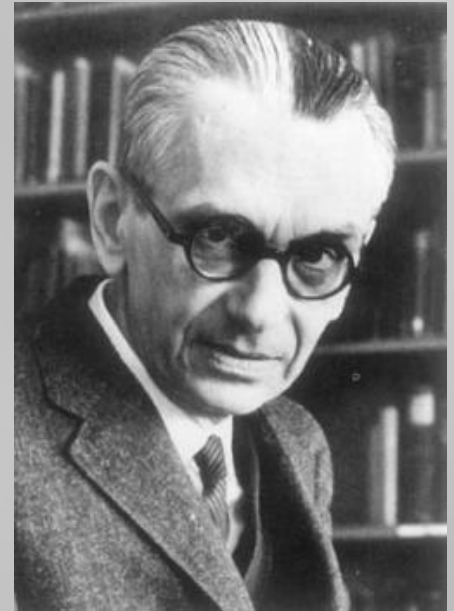
Can all problems be solved?



- Among Hilbert's 23 problems, 1903
- Problems 3, 7, 10, 11, 13, 14, 17, 19, 20, and 21 have a resolution that is accepted by consensus.
- Problems 1, 2, 5, 9, 15, 18+, and 22 have solutions that have partial acceptance, but there exists some controversy as to whether it resolves the problem.
 - (18 is Kepler conjecture, requires computer proof)
- 8: the Riemann hypothesis, 12: abstract number theory
- 4, 6, 16, and 23 are too vague to ever be described as solved.
- **2nd problem: Prove that the axioms of arithmetic are consistent.**
 - Continuing Leibniz dream

Can arithmetic be proven consistent? No!

- Gödel Incompleteness Theorems (1931)
- First: “Any effectively generated theory capable of expressing elementary arithmetic cannot be both **consistent** and **complete**.”
- Gödel's second incompleteness theorem, proved in 1931, shows that no proof of its consistency can be carried out within arithmetic itself.
 - “For any formal effectively generated theory T including basic arithmetical truths and also certain truths about formal provability, T includes a statement of its own consistency if and only if T is inconsistent.”
- Leads to Halting Problem in computer science (Turing, 1936)
 - Someday we'll talk about the Turing Hierarchy 😊





What can then be solved automatically?

- Independently: Alonzo Church and Alan Turing, 1936
- Formalized the notion of “decidable”, which is equivalent to “what is computable.”
- Church (1936) invented the “lambda calculus”, written λ -calculus.
 - A “calculus” is a method of moving symbols, not the undergrad course.
- Turing (1936/37) invented “Turing Machines”.
- Turing showed them to be equivalent.



Outcome

- Modern machines are based on Turing Machines (actually, von Neumann machines, allowing random memory access).
- Modern computer languages are based on concepts from lambda calculus, which allow telling such a machine what to do.
- Imperative languages (Pascal, C, Fortran) are based on the way Turing machine are instructed
- Functional languages like Miranda, Lisp, ML, Haskell, F#, are based on lambda calculus.

Reduction

- Functional program consists of
 - Expression E (represents both algorithm and input)
 - This is subject to rewrite rules
 - Reduction: replace a part P of E by some P' , Provided that $P \rightarrow P'$ is one of the rules

$$E[P] \rightarrow E[P']$$

Example:

- `first of (sort (append (`dog', `rabbit') (sort ((`mouse', `cat')))))`
- `->first of (sort (append (`dog', `rabbit') (`cat', `mouse')))`
- `->first of (sort (`dog', `rabbit', `cat', `mouse'))`
- `-first of (`cat', `dog', `mouse', `rabbit')`
- `-> `cat':`



Properties

- Church-Rosser property: normal form is independent of order of rewrite rules
- Rewrite rules are called combinators

λ -Calculus

- Two operations: application and abstraction
- Application
 - $F \cdot A$ or FA is applying data F as an algorithm to data A considered as input.
 - Theory is type-free, so FF makes sense.
- Abstraction
 - If $M[x]$ is expression containing x , then the function $\lambda x. M[x]$ represents the function $x \mapsto M[x]$
 - Example: $(\lambda x. 2x + 1)3 = 2 * 3 + 1 (= 7)$
- Called “the smallest programming language of the world”.
- Introduced in 1930 by Alonzo Church to formalize the notion of computability.

Abstraction

- Binds the free variable x in M , that is, $\lambda x. xy$ has x bound and y free.

Multiple arguments:

- $f(x, y)$ depends on 2 variables.
- Can define
- $F_x = \lambda y. f(x, y)$
- $F = \lambda x. F_x$
- Then $(Fx)y = F_x y = f(x, y)$
- Leads to association:
 - $FM_1M_2 \cdots M_n = (\cdots (FM_1) M_2) \cdots M_n$
- Called currying, after Haskell Curry
- Haskell computer language named after him

Notation

Definition 1:

λ -terms: the set Λ built from the infinite set of variables $V = \{v', v'', v''', \dots\}$ using application and abstraction:

$$\begin{aligned}x &\in V \Rightarrow x \in \Lambda \\M, N &\in \Lambda \Rightarrow (MN) \in \Lambda \\M &\in \Lambda, x \in V \Rightarrow (\lambda x M) \in \Lambda\end{aligned}$$

We use lowercase $x, y, z \dots$ for variables and caps $M, N, L \dots$ for Λ -terms

Notation

Definition 2:

The free variables in M , written $FV(M)$ is

$$FV(x) = \{x\}$$

$$FV(MN) = FV(M) \cup FV(N)$$

$$FV(\lambda x M) = FV(M) - \{x\}$$

A not-free variable is bound.

M is a closed λ -term (also called a combinator) if $FV(M) = \emptyset$

Definition 3:

The result of substituting N for the free occurrences of x in M is denoted

$$M[x := N]$$

Λ Calculus

- The principal axiom is

$$(\lambda x M)N = M[x := N]$$

- Logical rules and axioms:
 - Equality

$$M = M$$

$$M = N \Rightarrow N = M$$

$$M = N, N = L \Rightarrow M = L$$

- Compatibility

$$M = M' \Rightarrow MZ = M'Z$$

$$M = M' \Rightarrow ZM = ZM'$$

$$M = M' \Rightarrow \lambda x. M = \lambda x. M'$$

- Proof: if $M = N'$ is provable in the λ – calculus, write $\lambda \vdash M = N$

Standard Combinators

Name	Definition	Action on terms
Identity	$\mathbf{I} = \lambda x. x$	$\mathbf{I}M = M$
Left	$\mathbf{K} = \lambda xy. x$	$\mathbf{K}MN = M$
Right	$\mathbf{K}_* = \lambda xy. y$	$\mathbf{K}MN = N$
	$\mathbf{S} = \lambda xyz. xz(yz)$	$\mathbf{S}MNL = ML(NL)$

Used to solve equations

Reduction and Normality

- Three reduction types:
 - α reduction: renaming a bound variable, e.g., $\lambda x. x \rightarrow \lambda y. y$
 - β reduction: applying functions to arguments , e.g. $(\lambda z. z * 2)5 \rightarrow 5 * 2$
 - η reduction: *extensionality*, that is, two functions are the same if and only if they give the same value for all arguments.

Normal Form

- Expression cannot be “rewritten” any further.
- Not all expressions have this property:
 - Consider $\lambda x. xxx$. Applied to itself leads to (TODO – keeps expanding)
Type equation here.
- TODO – state it
- TODO – can always be done
- TODO – expand outermost evaluation first
- TODO – some expressions never terminate, have no normal form.

Arithmetic

- Define $F^0(M) \equiv M, F^{n+1}(M) \equiv F^n(M)$
- Church numerals: c_0, c_1, c_2, \dots defined via $c_n \equiv \lambda f x. f^n(x)$

$$\mathbf{A}_+ \equiv \lambda x y p q. x p (y p q)$$

- Operations: $\mathbf{A}_* \equiv \lambda x y z. x(yz)$

$$\mathbf{A}_{exp} \equiv \lambda x y. y x$$

- Then, for all nonnegative integers n, m

$$\mathbf{A}_+ c_n c_m = c_{n+m}$$

$$\mathbf{A}_* c_n c_m = c_{n*m}$$

$$\mathbf{A}_{exp} = c_{(n^m)}$$

- $\mathbf{A}_+ c_n c_m =$

Booleans

- **true** $\equiv \mathbf{K}$ ($\equiv \lambda xy. x$), **false** $\equiv \mathbf{K}_*$ ($\equiv \lambda xy. y$)
- If B can be considered a boolean (i.e., true or false) then
“If B then P else Q ” can be represented by BPQ
- Ordered pair: $[M, N] \equiv \lambda z. \text{if } z \text{ then } M \text{ else } N$ ($\equiv \lambda z. zMN$)

Y-Combinator

- Creates recursion from single steps
- It is a function Y such that for any function g , $Y(g)=g(Y(g))$.
- Y finds fixed-points for functions, $Y(g)=p$ where $p=g(p)$
- Does it exist? Yes, there are infinitely many.
 - Y-Combinator $Y = \lambda f.(\lambda x.f (x x)) (\lambda x.f (x x))$
 - Another for amusement:
 $Y_k = (L L)$ with
 $L = \lambda abcdefghijklmnopqrstuvwxyzr. (r (t h i s i s a f i x e d p o i n t c o m b i n a t o r))$

Y-Combinator

- Definition: $Y = \lambda f.(\lambda x.f (x x)) (\lambda x.f (x x))$. Apply to arbitrary function g
- $Y g = (\lambda f . (\lambda x . f (x x)) (\lambda x . f (x x))) g$ (by definition of Y)
- $= (\lambda x . g (x x)) (\lambda x . g (x x))$ (β -reduction of λf : applied main function to g)
- $= (\lambda y . g (y y)) (\lambda x . g (x x))$ (α -conversion: renamed bound variable)
- $= g ((\lambda x . g (x x)) (\lambda x . g (x x)))$ (β -reduction of λy : applied left function to right function)
- $= g (Y g)$ (by second equality)
- Repeat, giving $g (g (g (g (Y g))))$ as deep as desired... Recursion.

TODO

- Y-Combinator $Y = \lambda f.(\lambda x.f (x x)) (\lambda x.f (x x))$
- Church numerals, add, exponent,
- True, false, pairs
- Definability, recursive functions, all computable functions.
- Reduction

Types

- So far the λ -calculus is a type free theory
- Example, $\mathbf{I} = \lambda x. x$ can be applied to itself: \mathbf{II}
- Typed λ -calculus introduced in Curry(1934) and Church (1940).
- Type is an “thing” assigned to a term.
- Type may be thought of as a units “dimension”, prevents adding meters to mass.



Type benefits

- Allows easier checking for correctness
- Allows easier compilation and running (if type of an expression is arithmetical, can use ALU instead of generalized code).
- Implicit typing, in ML, assigned by compiler
- Explicit typing, common, assigned by programmer
- Not all computable functions can be represented by a typed term!



Typing Results

- Worked out in theory a long time ago
- Computer languages keep playing with various balances between the various flavors of typing.

C# Lambda Functions

- $(x, y) \Rightarrow x == y$
- ```
TestDelegate myDel = n => { string s = n + " " + "World";
Console.WriteLine(s); }; myDel("Hello");
```

# C++0x Lambda Functions

- `[](int x, int y) { return x + y; }`
- `std::vector<int> some_list; int total = 0; std::for_each(some_list.begin(), some_list.end(), [&total](int x) { total += x; });`



# F# Lambda Functions

# Bibliography

- [1] Introduction to Lambda Calculus, Henk Barendregt and Erik Barendsen, March 2000.
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