Space +Time = Space-time

Chris Lomont 2011

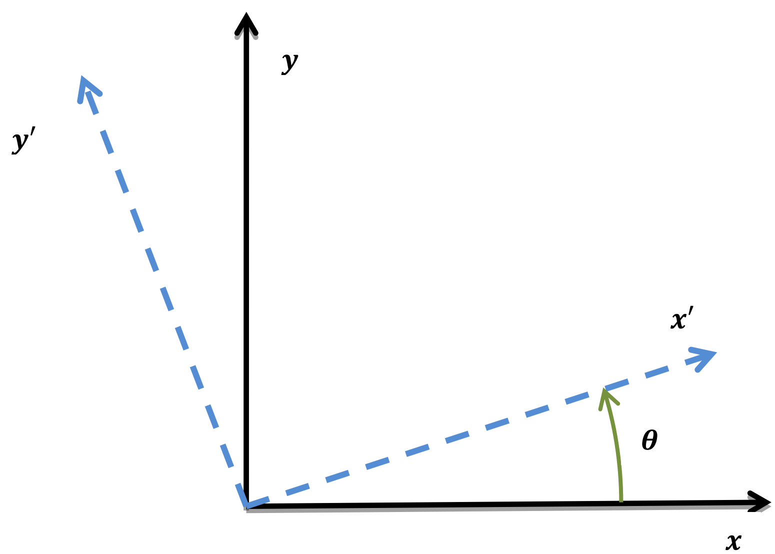
This talk presents a simple argument showing how time and space are mixed into what is called space-time. The main idea is to show that what one observer measures as space and time are mixed together in what a different observer (moving with respect to the first observer) measures as space and time. This difference is shown to be similar to how 2D x and y coordinates are rotated into each other during a normal 2D rotation.

Measurement

Two of the earliest items measured were length and time.

Cartesian coordinates (Rene Descartes, 17th century) allowed labeling points with numbers, allowing concise formulas to be developed.

Talk about measuring with a rod, then axes, giving xy-plane. Now points can be identified with an x, y coordinate pair. What if someone else picks a different set of axes?



One observer sets up one coordinate system P with coordinates x and y to make measurements. a second observer sets up a coordinate system P' with coordinates x' and $y'$to make measurements. For simplicity suppose each picks the same origin, but the angle θ between them can be nonzero.

Both observers use same definition of length (i.e., same measuring sticks). Given a point (x, y) in P how does this relate to a point (x', y') in P'?

The following transformation can be derived showing how they relate

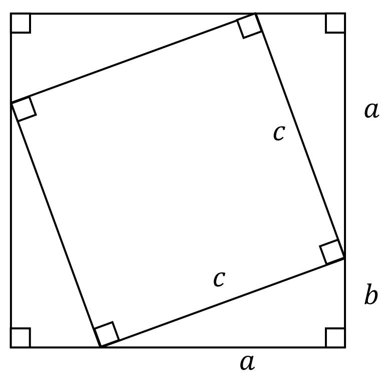
$$x' = \quad x\cos\theta \; + y\sin\theta$$
$$y' = -x\sin\theta \; + y\cos\theta$$

(One proof would be to write a point $Q = (a, b)$ in P and $Q' = (a', b')$ in P'. Both are the same distance d from the origin, and can be written as $Q = d(\cos\alpha, \sin\alpha)$ and $Q' = d(\cos\beta, \sin\beta)$ where $\beta = \alpha - \theta$ and then use trig angle formulas.)

Note that what one observer calls x' is a mixture of what another observer calls x and y. There is no ***inherent*** x and y in the universe. They are mixed together depending on the observer. The rest of this talk will show that time and space are also mixed in this manner, and the mixture depends on the observer.

Want them to agree on distances between points (x_1, y_1) to (x_2, y_2) in P and (x_1', y_1') to (x_2', y_2') in P'.

Recall the Pythagorean Theorem: for a right triangle with sides a, b, c with c the longest one, $a^2 + b^2 = c^2$. This gives a distance formula.



Writing the total area in 2 ways and expanding:

$$(a+b)^2 = 4\left(\frac{ab}{2}\right) + c^2$$
$$a^2 + 2ab + b^2 = 2ab + c^2$$
$$a^2 + b^2 = c^2$$

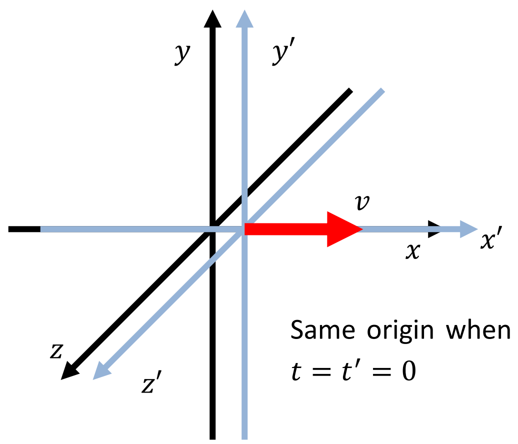
Letting $\Delta x = x_2 - x_1$, $\Delta y = y_2 - y_1$, $\Delta x' = x_2' - x_1{}'$, and $\Delta y' = y_2' - y_1{}'$, the Pythagorean Theorem gives that $d^2 = (\Delta \mathrm{x})^2 + (\Delta y)^2$ and $d'^2 = (\Delta \mathrm{x}')^2 + (\Delta y')^2$. We can check plugging in the transform leaves this value $d^2 = d'^2$ the same; it is called an *invariant* of the transformation.

Galilean transformation

Next we want to incorporate time into the transformation. Like length, time is defined with a unit periodic phenomenon, and the passage of time is measured in multiples of the periodic event. Early periodic events were astronomical, such as the length of a day or length of a year, a water clock, or a pendulum. More recent precise periodic events are used in the definition of the second. Since 1967 the second has been defined to be the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom.

With length and time defined, we can define *velocity* as change in position/change in time.

To understand how space and time interact, we need two observers in motion relative to one another. When one observer moves at a constant velocity relative to the other observer, each frame is called an *inertial frame*. The reason this type of motion is useful to analyze is that objects moving with constant velocity from the viewpoint of one observer will be seen the same way by the other observer (although with perhaps a different direction and rate, which depend on the coordinate systems chosen). The main point is two inertial frames differ by at most a constant velocity.



Let two observers in reference frames R and R' use their own Cartesian coordinate systems to measure space and time. R uses (t, x, y, z) and R' uses (t', x', y', z'). For simplicity, assume the coordinate systems are oriented so the x and x' axes are collinear and oriented in the same direction, the y and y' axes are parallel and oriented in the same direction as are the z and z' axes. Let the (space) origin in R' move at a constant velocity v along the x axis in the positive direction. Note both R and R' measure the same velocity v. Finally, suppose when the (space) origins overlap the times are $t = t' = 0$.

Now how do observers in the two frames convert from one system of coordinates to the other? It seems clear that $y = y'$ and $z = z'$. Simple (but ultimately physically incorrect) it seems $t = t'$, and $x' = x - vt$.

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

Let's consider $x' = x - vt$. This can be checked by considering each space origin as viewed by the other frame.

Velocities additive: suppose an observer on the ground measures a runner speed going by at 15 mph. Suppose the runner passes a walker going 5 mph. What speed would the walker measure for the runner, relative to the walker? Simple intuition seems to say the walker would measure 15mph-5mph = 10mph faster. This says velocities are additive, $v_T = v_1 + v_2$, and agrees with intuition. We can check this from the transformation:

Differentiating $x' = x - vt$ with respect to t' and noting $\frac{dt'}{dt} = 1$ gives

$$v_2 = \frac{dx'}{dt'} = \frac{dx}{dt}\frac{dt}{dt'} - v_1\frac{dt}{dt'} = v_T - v_1$$
$$v_T = v_1 + v_2$$

This is how physicists thought the world worked until the late 1800's when Maxwell's equations for electromagnetism did not transform in this manner between observers.

Speed of light

There was a huge incentive to measure the speed of light accurately since the 1600s, since Ole Rømer in 1676 demonstrated the speed was finite by studying the apparent motion of Jupiter's moon Io. Knowing the speed would allow humankind to know the distances to other planets, the size of the solar system, and many other astronomical values, and thus was of immense scientific interest.

One experiment wanted to measure Earth's "absolute" movement through space, and it was assumed we moved through an invisible "water" that transmitted light, called the *luminiferous ether*. Thus it was reasoned light would move at different velocities depending on how it faced the current in this "water". Light beams moving in perpendicular directions should be able to detect this difference in drift, especially if measured as the Earth moved around the sun. So for many years very careful experiments were done to detect this expected difference in the speed of light depending on direction of the Earth's movement around the sun. No matter how clever the experiments were, no one could find the expected variance.

Thus was the first empirical evidence that the speed of light is the same, no matter what the state of the observer. This constant speed of light is *exactly* $c = 299{,}792{,}458$ meters per second (as a result of defining the meter in terms of c). This is approximately 186,282 miles per second.

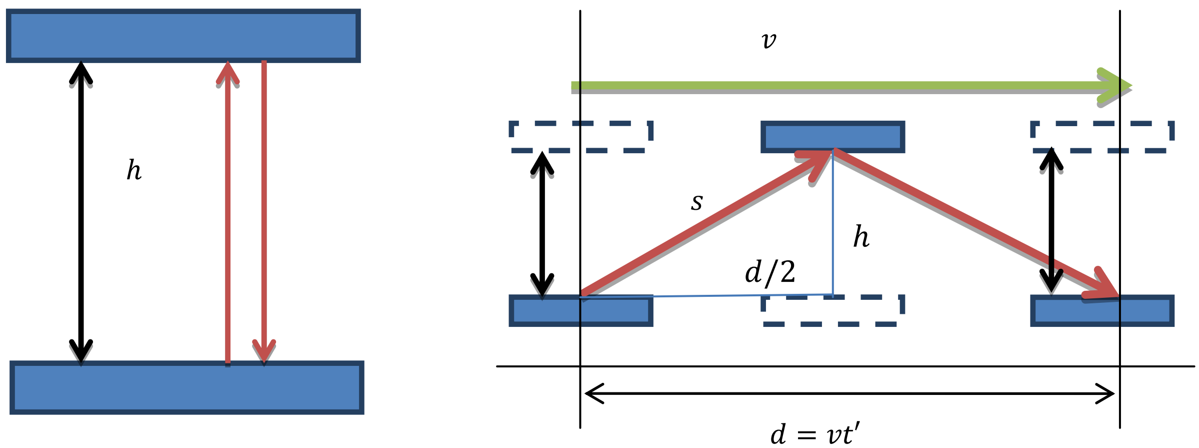
Thus if you are on a spaceship moving at half the speed of light $c/2$ relative to a planet, and the planet shoots a light beam that passes you, and you measure its speed as it passes, you do NOT measure the "remaining" half of the speed of light; you measure the constant speed of light c. However, the planet measures the speed of light to be c also. This means, at least for light, you cannot add velocities and thus the Galilean view of space and time is *wrong*. For example, by approaching a light source rapidly you do not measure the light as going faster than if you stood still relative to the light source.

Since the Galilean view is based on time, length, and velocity (which is length/time), and these do not add up nicely (at least for light), something has to differ from our intuition.

Light Clocks

Now we do a thought experiment. What happens if we assume the speed of light is the same value no matter what inertial frame an observer is in?

A simple clock is built that consists of two mirrors, distance h apart. A lightbeam is fired from the lower mirror, bounces off the top mirror, and returns in time t. This gives $2h = ct$.



Suppose this clock is stationary and an observer moves to the *left* at velocity v relative to the stationary clock. From his viewpoint the clock is moving right. What path does he see the light move? In a diagonal direction, first diagonal up and then down where the beam hits the bottom sensor.

What happens qualitatively? Suppose the light clock always registers one second per tick for the observer with the clock (for a very large h). To register one tick in the moving frame the light has to travel farther, so it takes more time as measured in the moving frame. Thus the moving observer might measure 10 seconds (ticks), but would see the light clock only register 1 tick. Thus, *from the viewpoint of the moving observer, the clock runs slow!* Conversely, if both observers had light clocks, each would see *the other clock run slow*! How can this be? It is similar to how if Bob sees Sue at a distance: Sue appears small to Bob and Bob appears small to Sue.

The moving observer sees the light move over a longer path, with each diagonal distance s being given by the Pythagorean Theorem:

$$s^2 = \left(\frac{d}{2}\right)^2 + h^2$$

The moving observer watches the light move in the path drawn, taking time t' from the observer viewpoint to return to the bottom sensor. The observer thus sees the light beam traverse distance $2s$ at speed c so measures the time elapsed as $2s = ct'$. During this time the light-clock moves a distance $d = vt'$.

Recall the observer riding with the clock thinks time is defined by $2h = ct$. Plugging in the time terms for the distances gives the relation (which we then use to isolate t')

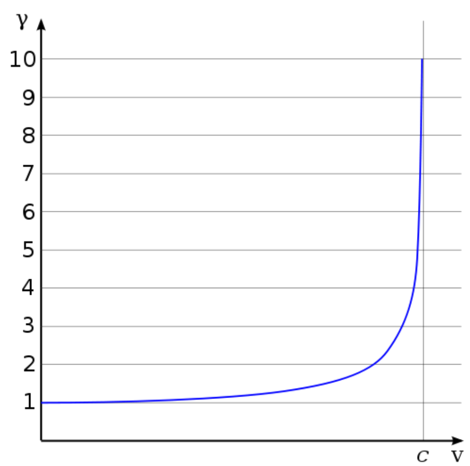
$$\left(\frac{ct'}{2}\right)^2 = \left(\frac{vt'}{2}\right)^2 + \left(\frac{ct}{2}\right)^2$$
$$\frac{c^2t'^2}{4} - \frac{v^2t'^2}{4} = \frac{c^2t^2}{4}$$
$$t'^2\left(1 - \frac{v^2}{c^2}\right) = t^2$$
$$t' = \frac{t}{\sqrt{1 - v^2/c^2}}$$

We took positive roots on each side so both see increasing time on the clocks. For simplicity define the Lorenz Factor $\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$. Then the times measured by the two observers are related by $t' = \gamma t$, that is, the number $t' > t$ for all velocities less than the speed of light.

Note this matches the Galilean transform $t' = t$ for small v values, where v/c is almost zero.

To recreate the qualitative example, setting $t' = 10$ and $t = 1$ gives approximately $v = 0.994987\ c =$ 185,348 miles per second. Pretty fast to see such an effect.

To better understand what this means, let's look at γ.



Factor	Ratio of v to c	mph
1	0	0
2	0.86602540	161325
3	0.94280904	175628
4	0.96824584	180367
5	0.97979590	182518
6	0.98601330	183677
7	0.98974332	184371
8	0.99215674	184821
9	0.99380799	185129
10	0.99498744	185348
100	0.99995000	186273
1000	0.99999950	186282

Notice also that v cannot be greater than c, otherwise the conversion factor becomes imaginary. This is where the "you cannot go faster than the speed of light" comes from.

What does this mean? Explain….

So this works for light clocks, but what about other clocks? Suppose some local clock differed from the light clock. Since the speed of light is constant for all observers, the observer with the mismatched local clock, by measuring a different time must also measure a different length. However this means length for the same observer depends on what type of clock he uses, which is not empirically true: a distance remains the same for the same observer no matter how it is measured. Thus all clocks must record the same time, and the effect $t' = \gamma t$ holds for all clocks.

Thus time is not the same for everyone! Why don't we see it? Using c = 670,616,629 mph, 31,556,926 seconds in a year, you have to go at speed 168,827 mph for one year to be 1 second different than a stationary clock.

So what experimental evidence is there?

Joseph Hafele and Richard Keating in October 1971 flew atomic clocks aboard commercial airliners twice around the world, once eastward and once westward. special relativity predicted -184 ± 18 nanosecond drift eastward and 96 ± 10 ns drift westward, and experiment (which also needed general relativity effects) demonstrated clock drift in accordance with the predictions.

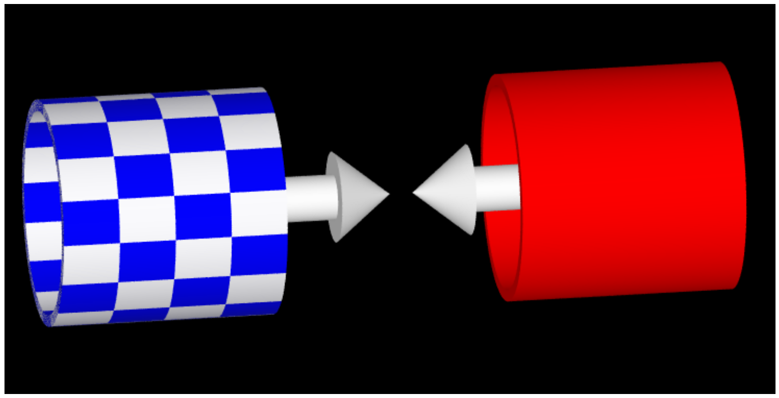
Another experiment measures muon lifetime, which averages 2.2 microseconds. When sped up their lifetime increases (their internal "clock" is slow compared to stationary observers) the factors predicted by special relativity. This was first done by Rossi-Hall in 1941 who measured muons going 99.94% the speed of light. These muons should decay in the upper atmosphere but reach the earth, their lifetimes extended by a factor of nine.

A final application proving special relativity works is in GPS. There are two effects. Time runs quicker for moving satellites (due to special relativity) by $7\,\mu s$/day. Time runs slower for the satellite in lower gravity (due to general relativity) by $45\,\mu s$/day. For GPS to work, it needs timing accuracy on the order of nanoseconds, and that 7 μs/day corresponds $7{,}000$ nanoseconds which is a drift in position of about 10km/day. The 7 μs lag can be easily check yourself, and it is done in an Appendix.

Lorenz Transform

So, if time is weird, but the velocity of light is constant for all observers, and velocity is distance/time, doesn't this mean distances act weird also? Let's study coordinate systems involving space and time coordinates. We start with inertial systems R with coordinates (x, y, z, t) and R' with coordinates (x', y', z', t').

We can make a lattice of clocks and cameras, all synched, by placing clocks at known distances from the origin in each frame, and synchronizing each frame using a light flash (say at noon) at the origin. Since each clock/camera knows its distance from the origin, and it knows the flash was scheduled at noon, and it knows the speed of light (assumed constant for all observers), each clock/camera now has a synchronized time in each frame.



With the motion only in the x direction, consider two hollow cylindrical shells (pictured) flying towards each other (one in each observer frame). Let one cylinder have a checkerboard pattern and one have a solid color. If the lengths in either the y or z directions between the observers, then one could set up a cylinder (or perhaps similar shape) such that one observer would see the checkerboard pattern cylinder enter the solid cylinder, and other observer would see the opposite. Since this would violate the observed consistency of the observations, it must mean that both observers agree on all distances in their respective yz and $y'z'$ planes. Thus $y' = y$ and $z' = z$, just as in the Galilean transformation.

So the only changes can be in x and t. It can be shown (see the appendix) that the transformation must be linear in the x and t coordinates, so we investigate the most general form of them:

$$x' = ax + bt$$
$$t' = ex + ft$$

Note $a = 1, b = -v, e = 0, f = 1$ is old Galilean transform, so it is a generalization of that transform.

The light clock example above shows the behavior of clocks at each origin, that is, for $x = x' = 0$ the relationship $t' = \gamma t$ must hold. This gives

$$t' = 0 + ft = \gamma t$$

so $f = \gamma$. (Recall $= \frac{1}{\sqrt{1-v^2/c^2}}$).

Suppose a red ball is kept at the space origin of R and a blue one is at the space origin of R'.

If R' watches the red ball in R there is a correspondence $(0, t)$ to $(-vt', t')$. This gives

$$-vt' = 0 + bt$$
$$t' = 0 + ft$$

Solving the first for t' and setting $ft = t' = -\frac{b}{v}t$ and noting this is true for all t, this gives $b = -vf = -v\gamma$.

If R watches the blue ball in R' there is a correspondence (vt, t) to $(0, t')$. This gives

$$0 = avt + bt$$
$$t' = evt + ft$$

This is true for any value of t so $av = -b = -(-v\gamma)$ simplifying to $a = \gamma$.

If a flashbulb goes off at $t = t' = x = x' = 0$ then both frames see the light speeding along their own x-axis. The wave front of the flash moves at speed c in both frames, so $x = ct$ corresponds to $x' = ct'$. Then

$$ct' = act + bt$$
$$t' = ect + ft$$

Solving the top for t' and setting them equal gives

$$t' = \left(a + \frac{b}{c}\right)t = (ec + f)t$$

Noting this is true for all t allows dividing out t, gives

$$a + \frac{b}{c} = ec + f$$

Solving for e and putting in the values found for $a = \gamma$, $b = -v\gamma$, and $f = \gamma$,

$$ec + \gamma = \gamma - \frac{v\gamma}{c}$$
$$e = -\frac{v\gamma}{c^2}$$

This gives for the transforms (called the Lorenz Equations):

$$t' = \gamma(t - vx/c^2)$$
$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$

As a sanity check, note when v is much smaller than c that γ is very close to 1, and these equations become the Galilean transformation. So the Lorenz equations are a correction to the Galilean transformation.

Finally let's interpret these equations and check experimental evidence.

Space + Time = Space-time

Now, consider only the x and t coordinates. Recall the 2D rotation

$$x' = \quad x\cos\theta \; + y\sin\theta$$
$$y' = -x\sin\theta \; + y\cos\theta$$

where one observer x value is a (linear) combination of another observers x and y. Now look at the Lorenz equations, expanded out a little, with grouping to make the match to a 2D rotation a little clearer:

$$x' = \qquad (\gamma)\,x + (-v\gamma)\,t$$
$$t' = -\left(\frac{v\gamma}{c^2}\right)x + \quad (\gamma)\,t$$

and note the same behavior: ***one observer has their time and space coordinates a mixed function of another observers time and space coordinates!*** In each case the new coordinates are a "simple" combination of some constants and the old coordinates.

Note that how one observer measures t' is a function of the how the other observer measures x and t, and similarly for x'. This intertwining of space and time gives a new construct called space-time. In words this means the concepts of space and time and how they are measured are dependent on observer, just like the rotation of coordinate planes show x and y are observer dependent.

Notice that no observer velocity allows time and space to "rotate" completely into one another. It only mixes them up.

To make this even more apparent, let $\beta = v/c$ and set $e^{\phi} = \gamma(1+\beta) = \sqrt{\frac{1+v/c}{1-v/c}}$. Then $e^{-\phi} = \gamma(1-\beta)$ and using hyperbolic cosine $\cosh\phi = \frac{e^{\phi}+e^{-\phi}}{2}$ and hyperbolic sine $\sinh\phi = \frac{e^{\phi}-e^{-\phi}}{2}$ some algebra gives that (scaling t to ct to make the equations cleaner)

$$x' = x\cosh\phi - ct\sinh\phi$$
$$ct' = -x\sinh\phi + ct\cosh\phi$$

This shows time and space are a "hyperbolic" rotation of axes just like x and y are a 2D rotation of axes.

Note the transformations can be applied to general inertial frames even if the direction is not on the x-axis and origins and times do not align.

Here are some experiments backing the constant speed of light, time dilation, and the Lorenz transform.

- *Chen et al., "Experimental Test of the Isotropy of Two-way Light Speed", A.S.N.U. Peking, 33, no. 5, pg 595 (1997).*An experiment similar to Brillet and Hall, with a limit of 1×10^{-18} in the anisotropy of *c* when performing experiments similar to Michaelson and Morley.

- *Luo et al., "New Experimental Limit on the Photon Rest Mass with a Rotating Torsion Balance", Phys. Rev. Lett, 90, no. 8, 081801 (2003).* A limit of 1.2×10^{-51} g (6×10^{-19} eV/c^2).

- *Schaefer, Phys. Rev. Lett.* ***82*** *no. 25 (1999), pg 4964.* For photons of 30 keV and 200 keV the speed of light is the same within a few parts in 10^{21}.

- *Essen and Froome, The Velocity of Light and Radio Waves (1969).* For frequencies between 10^8 and 10^{15} Hz the speed of light is constant within 1 part in 10^5.

- *Hafele and Keating, Nature* ***227*** *(1970), pg 270 (proposal).Science Vol.* ***177*** *pg 166–170 (1972) (experiment).* They flew atomic clocks on commercial airliners around the world in both directions, and compared the time elapsed on the airborne clocks with the time elapsed on an earthbound clock (USNO). Their eastbound clock lost 59 ns on the USNO clock; their westbound clock gained 273 ns; these agree with GR predictions to well within their experimental resolution and uncertainties (which total about 25 ns). By using four cesium-beam atomic clocks they greatly reduced their systematic errors due to clock drift.

- *Sherwin, "Some Recent Experimental Tests of the 'Clock Paradox'", Phys. Rev.* ***129*** *no. 1 (1960), pg 17.* He discusses some Mössbauer experiments that show that the rate of a clock is independent of acceleration (~10^{16} g) and depends only upon velocity.

Conclusion

This note has shown from basic arguments how space and time get intermixed once one assumes the speed of light is the same for all observers. The Lorenz transformation is derived from simple principles, without resorting to math beyond algebra (with a little calculus for illustration). Experimental evidence shows the applicability and correctness of time dilation and the Lorenz transform as far as has been measured. In particular, the old Galilean view of the world is wrong.

Appendix

Linearity of Transform

This is a short proof that the mapping between inertial frames has to be linear.

Let L be the mapping between inertial frames R and R' in standard position as described earlier. L fixes $y = y'$ and $z = z'$ for the reasons explained earlier. So it is enough to consider $L(x,t) = (x',t')$. Write $A = (x_A, t_A)$ and $L(A)$ for shorthand.

Pick any two points A and B in R. These can be treated as vectors from the origin in R.

L maps lines to lines (that is, things with constant velocity in one frame have constant velocity in the other, else one frame sees a "force" the other does not). L maps parallel lines to parallel lines (other wise items with same velocity in one frame have differing velocities in the other).

Now complete the parallelogram from A and B (as vectors). L maps this parallelogram to one in the moving frame, and maps $A + B$ to $L(A + B)$ which, by completion of the parallelogram in the moving frame, gives $L(A + B) = L(A) + L(B)$. Repeating with $-B$ gives $L(A - B) = L(A) - L(B)$.

Replacing B with $(n-1)A$ for any integer n and inducting gives $L(nA) = L\big((n-1)A + A\big) = (n-1)L(A) + L(A) = nL(A)$. For integers p, q this gives $qL\left(\frac{p}{q}A\right) = L\left(q\frac{p}{q}A\right) = L(pA) = pL(A)$. Dividing by q gives $L\left(\frac{p}{q}A\right) = \frac{p}{q}L(A)$.

Now, for any real value r, take a sequence of rational numbers $\frac{p_i}{q_i} \to r$. Note $L\left(\frac{p_i}{q_i}A - rA\right) = \frac{p_i}{q_i}L(A) - L(rA)$ for each i. As $i \to \infty$, the left hand side goes to $L(0) = 0$ and the right hand side goes to $rL(A) - L(rA)$. L has to map close points in R to close points in R' otherwise velocities in one frame converging to the same value would diverge in the other frame, violating consistency in physics (both observers should agree on objects going the same velocity). Since the two sides are equal for every value of i, and the left side converges to 0, they are equal in the limit, giving $0 = rL(A) - L(rA)$ which gives $L(rA) = rL(A)$ for all real values r.

Now let $L(1,0) = (a,e)$ and $L(0,1) = (b,f)$. Then $L(x,t) = L\big(x(1,0) + t(0,1)\big) = xL(1,0) + tL(0,1) = x(a,e) + t(b,f) = (ax + bt, ex + ft)$ and L must be of the form

$$x' = ax + bt$$
$$t' = ex + ft$$

QED.

GPS Time Lag

To check the GPS satellite time lag of 7 μs, a few values are needed. These are the mass of the earth $M = 5.24 \times 10^{24}$kg, the (average) radius of the GPS satellite orbits from the center of the earth $R = 26560$km$= 2.65 \times 10^{7}$m, the speed of light $c = 299792458$m/s, and the gravitational constant

$G = 6.673 \times 10^{-11} \frac{m^3}{kg\ s^2}$. Letting the mass of the satellite be m and the velocity of the satellite be v, the gravitational force on it is $F = G\frac{mM}{R^2} = ma$ and the balancing force to keep it in orbit is the centripetal force $F = \frac{mv^2}{R} = ma$. Setting these equal gives

$$G\frac{mM}{R^2} = \frac{mv^2}{R}$$

$$v = \sqrt{\frac{GM}{R}}$$

Plugging in the values gives $v = 3872$ m/s. There are 86400 seconds in a day, which is $8.64 \times 10^{10} \mu s$. Computing the time lag gives

$$7 + 8.64 \times 10^{10} = 8.64 \times 10^{10} / \sqrt{1 - 3872^2/299792458^2}$$

Note the extra $7\mu s$ on the left hand side. So even a naïve computation of the time dilation gives the correct answer (presumably computed by GPS engineers using very careful methods accounting for orbit shape, variations in speed at different parts of the orbit, curvature effects, and other factors).