Solution to a hard integral problem

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1 The problem

For a very challenging integration problem, solvable with 3rd semester calculus, look to the textbook, "*Calculus and Analytic Geometry*", 6th edition, by George B. Thomas and Ross L. Finney. The following problem appears in section 16.7 on multiple integration:

37. WARNING: Hard problem. Setting up the integral is straightforward, but integrating the result takes hours. (It took MACYSMA 20 minutes.)

A square hole of side length $2b$ is cut symmetrically through a sphere of radius a ($a > b\sqrt{2}$). Find the volume removed.

2 The solution

Using symmetry, we compute over a triangular region $0 \le x \le b$, $0 \le y \le x$, with height $\sqrt{a^2 - x^2 - y^2}$ to get

$$V = 16 \int_0^b \int_0^x \sqrt{a^2 - x^2 - y^2}\, dy\, dx \tag{1}$$

Let $y = \sqrt{a^2 - x^2} \sin\theta$, so $dy = \sqrt{a^2 - x^2} \cos\theta\, d\theta$, and use the identity $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ to get

$$\begin{aligned}
V &= 16 \int_0^b \int_0^\alpha \left(\sqrt{a^2 - x^2}\right)^2 \sqrt{1 - \sin^2\theta} \cos\theta\, d\theta\, dx \\
&= 16 \int_0^b \int_0^\alpha \left(a^2 - x^2\right) \cos^2\theta\, d\theta\, dx && (2) \\
&= \frac{16}{2} \int_0^b \int_0^\alpha \left(a^2 - x^2\right)(1 + \cos 2\theta)\, d\theta\, dx && (3) \\
&= 8 \int_0^b \left(a^2 - x^2\right) \left(\theta + \frac{1}{2}\sin 2\theta\right)\Bigg|_0^\alpha dx && (4)
\end{aligned}$$

where we define $\alpha = \sin^{-1} \frac{x}{\sqrt{a^2 - x^2}}$. Now using $\frac{1}{2}\sin 2\theta = \sin\theta\cos\theta$, evaluate at $\theta = \alpha$; i.e., $\sin\theta = \frac{x}{\sqrt{a^2 - x^2}}$, $\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \frac{x^2}{a^2 - x^2}}$ (true for quadrant I), giving

$$\begin{aligned}
\left(\frac{1}{2}\sin 2\theta\right)\Bigg|_0^\alpha &= \frac{x}{\sqrt{a^2 - x^2}} \sqrt{1 - \frac{x^2}{a^2 - x^2}} - 0 && (5) \\
&= \frac{x\sqrt{a^2 - 2x^2}}{a^2 - x^2} && (6)
\end{aligned}$$

Substituting expression 6 into equation 4 gives

$$\begin{aligned}
V &= 8 \int_0^b \left(a^2 - x^2\right) \left(\alpha + \frac{x\sqrt{a^2 - 2x^2}}{a^2 - x^2}\right) dx && (7) \\
&= 8 \int_0^b \left(a^2 - x^2\right) \sin^{-1} \frac{x}{\sqrt{a^2 - x^2}}\, dx + \\
&\quad 8 \int_0^b x\sqrt{a^2 - 2x^2}\, dx && (8) \\
&= I_1 + I_2 && (9)
\end{aligned}$$

where we relabeled the two integrals in equation 8 as the two integrals $I_1 = 8 \int_0^b \left(a^2 - x^2\right) \sin^{-1} \frac{x}{\sqrt{a^2 - x^2}}\, dx$ and $I_2 = 8 \int_0^b x\sqrt{a^2 - 2x^2}\, dx$.

We evaluate I_2 first, since it is easier. Substitute $u = a^2 - 2x^2$, so $du = -4x\ dx$ and $x\ dx = -\frac{1}{4}\ du$, giving

$$I_2 = 8\int_0^b \sqrt{a^2-2x^2}\cdot x\ dx \quad (10)$$

$$= 8\int_{a^2}^{a^2-2b^2} \sqrt{u}\left(-\frac{1}{4}\right)\ du \quad (11)$$

$$= -2\int_{a^2}^{a^2-2b^2} u^{\frac{1}{2}}\ du \quad (12)$$

$$I_2 = \frac{4}{3}\left(a^3 - \left(a^2-2b^2\right)^{\frac{3}{2}}\right) \quad (13)$$

Now to do the more difficult I_1, which we do by parts by choosing u and dv and using $\int u\ dv = uv - \int v\ du$

$$u = \sin^{-1}\frac{x}{\sqrt{a^2-x^2}}, \qquad v = a^2x - \frac{x^3}{3}$$

$$du = \frac{a^2}{(a^2-x^2)\sqrt{a^2-2x^2}}\ dx, \quad dv = (a^2-x^2)\ dx$$

Thus I_1 becomes

$$I_1 = 8\sin^{-1}\frac{b}{\sqrt{a^2-b^2}}\left(a^2b - \frac{b^3}{3}\right) + I_3 \quad (14)$$

where $I_3 = -8\int_0^b \frac{a^2\left(a^2x-\frac{x^3}{3}\right)}{(a^2-x^2)\sqrt{a^2-2x^2}}\ dx$. So now we are left with only I_3 to evaluate. Substituting $u = \sqrt{a^2-2x^2}$, then $\left(-\frac{1}{2}\right)\ du = \frac{x}{\sqrt{a^2-2x^2}}\ dx$, and $x^2 = \frac{a^2-u^2}{2}$, so

$$I_3 = -8\int_0^b \frac{a^2\left(a^2-\frac{x^2}{3}\right)}{(a^2-x^2)}\cdot\frac{x}{\sqrt{a^2-2x^2}}\ dx \quad (15)$$

$$= -\frac{-8a^2}{2}\int_a^{\sqrt{a^2-2b^2}} \frac{\left(a^2-\frac{a^2-u^2}{6}\right)}{\left(a^2-\frac{a^2-u^2}{2}\right)}\ du \quad (16)$$

$$= \frac{4a^2}{3}\int_a^{\sqrt{a^2-2b^2}} \frac{u^2+5a^2}{u^2+a^2}\ dx \quad (17)$$

$$= \frac{4a^2}{3}\int_a^{\sqrt{a^2-2b^2}} 1 + \frac{4a^2}{u^2+a^2}\ dx \quad (18)$$

$$I_3 = \frac{4a^2}{3}\left(\sqrt{a^2-2b^2} - a\right) + I_4 \quad (19)$$

with $I_4 = \frac{16a^4}{3}\int_a^{\sqrt{a^2-2b^2}} \frac{1}{u^2+a^2}\ dx$. Substituting $u = a\tan\phi$, $du = a\sec^2\phi\ d\phi$ in I_4, and using the identity $1+\tan^2\phi = \sec^2\phi$ results in

$$I_4 = \frac{16a^4}{3}\int_a^{\sqrt{a^2-2b^2}} \frac{1}{u^2+a^2}\ dx \quad (20)$$

$$= \frac{16a^4}{3}\int_{\frac{\pi}{4}}^{\tan^{-1}\frac{\sqrt{a^2-2b^2}}{a}} \frac{1}{a^2+a^2\tan^2\phi} a\sec^2\phi\ d\phi \quad (21)$$

$$= \frac{16a^4}{3}\int_{\frac{\pi}{4}}^{\tan^{-1}\frac{\sqrt{a^2-2b^2}}{a}} \frac{a}{a^2}\cdot\frac{\sec^2\phi}{1+\tan^2\phi}\ d\phi \quad (22)$$

$$= \frac{16a^3}{3}\int_{\frac{\pi}{4}}^{\tan^{-1}\frac{\sqrt{a^2-2b^2}}{a}} \frac{\sec^2\phi}{\sec^2\phi}\ d\phi \quad (23)$$

$$I_4 = \frac{16a^3}{3}\left(\tan^{-1}\frac{\sqrt{a^2-2b^2}}{a} - \frac{\pi}{4}\right) \quad (24)$$

Summing up the results in equations 9, 13, 14, 19, and 24, and simplifying gives the final answer to the problem

$$V = 8\left(a^2b - \frac{b^3}{3}\right)\sin^{-1}\frac{b}{\sqrt{a^2-b^2}} + \frac{8b^2}{3}\sqrt{a^2-2b^2} + \frac{16a^3}{3}\left(\tan^{-1}\frac{\sqrt{a^2-2b^2}}{a} - \frac{\pi}{4}\right) \quad (25)$$

Note this has the correct behavior that as $b \to 0$, $V \to 0$, and that all terms have cubic dimension, giving a volume as expected. Also, as $b \to \frac{a}{\sqrt{2}}$, the volume not removed is four circular caps, whose volume is easy to find, leaving $V = \pi a^3\left(\frac{5\sqrt{2}-4}{3}\right)$, as expected.

Finally, the answer in the back of the textbook,

$$V = \frac{4\pi a^3}{3} + \frac{8b\left(3a^2-b^2\right)}{3}\sin^{-1}\frac{b}{\sqrt{a^2-b^2}} + \frac{8b^2}{3}\sqrt{a^2-2b^2} - \frac{16a^3}{3}\tan^{-1}\frac{a}{\sqrt{a^2-2b^2}} \quad (26)$$

although seemingly different than equation 25, also satisfies these two limiting cases.